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The slow translation of a sphere in a rotating viscous fluid

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The motion of a sphere along the axis of rotation of an incompressible viscous fluid that is rotating as a solid mass is investigated by means of numerical methods for small values of the Reynolds and Taylor numbers. The Navier–Stokes equations governing the steady axisymmetric flow can be written as three coupled, nonlinear, elliptic partial differential equations for the stream function, vorticity and rotational velocity component. Two numerical methods are employed to solve these equations. The first is the method of series truncation in which the dependent variables are expressed as series of orthogonal Gegenbauer functions and the equations of motion are then reduced to three coupled sets of ordinary differential equations, which are integrated numerically subject to their boundary conditions. In the second method, specialized finite-difference techniques of solution are applied to the two-dimensional partial differential equations. These techniques employ finite-difference equations with coefficients that depend upon the exponential function; a particularly suitable form of approximation for use in calculating numerical solutions is obtained by expanding the exponential coefficients in powers of their exponents.

Calculated results obtained by the two methods are in good agreement with each other. The calculations have been carried out according to theoretical assumptions that simulate the experiments of Maxworthy (1965) in which the sphere experiences no resultant torque exerted by the surrounding fluid and is free to rotate with constant angular velocity. Numerical estimates of this angular velocity and of the drag exerted by the fluid on the sphere are found to agree well with the experimental results for Reynolds and Taylor numbers in the range from zero to unity. The results for small values of the Reynolds number are also consistent with theoretical work of Childress (1963, 1964) which is valid as the Reynolds number tends to zero.

1. Introduction

The motion created by a sphere moving along the axis of rotation of a rotating fluid is of great interest since in this flow two kinds of basic motion, rotation and translation, interact and modify each other. Many novel features are introduced into the possible types of motion of a fluid due to the effects of rotation, as indicated by

Greenspan (1968*b*) and Maxworthy (1970). In a viscous fluid the flow depends upon two dimensionless parameters, namely the Ekman number $E = \nu/\omega_0 a^2$ and the Rossby number $R_0 = U/2\omega_0 a$, where a and U denote the radius and the velocity of the sphere respectively, ω_0 is the angular velocity of the mass of rotating fluid and ν is the coefficient of kinematic viscosity of the fluid. The reciprocal of the Ekman number is the Taylor number $T = a^2\omega_0/\nu$ and the ratio $2R_0/E$ is the usual Reynolds number $R = Ua/\nu$. In the present calculations we have concentrated on small values of R and T in the range from zero to unity, which in essence is the range covered by the experiments of Maxworthy (1965).

Taylor (1921) observed experimentally that, when a sphere is allowed to move slowly through a fluid that is in a state of solid-body rotation, a column of fluid is pushed ahead of the sphere like a solid mass having zero axial velocity relative to the moving body. This phenomenon is now known as the Taylor column and was predicted theoretically by Proudman (1916). Taylor reported that this columnar type of regime appears for values of $1/R_0$ above about 6. Further observations by Long (1953) demonstrated the existence of a train of waves downstream of a conical body with a hemispherical front surface at small values of $1/R_0$. When $1/R_0$ approached a value of about 6, a strong cyclonic vortex behind and a Taylor column ahead of the body was observed. In order to explain the essential features of this type of flow there have been numerous theoretical studies of the problem, for example by Taylor (1917, 1922), Grace (1926), Stewartson (1952, 1958, 1968), Morrison & Morgan (1956), Moore & Saffman (1968, 1969), Miles (1969), Barnard & Pritchard (1975) and Hocking, Moore & Walton (1979).

The uniform slow motion created by a sphere moving along the axis of rotation of a viscous fluid has been examined theoretically by Childress (1964) for small values of R and T. By means of singular perturbation techniques he obtained a solution which satisfies the no-slip and zero-torque conditions on the sphere and also the conditions valid far from the sphere, namely that the fluid rotates with constant angular velocity about the axis of translation of the sphere. The problem was finally solved by matching techniques and an analytical expression for the correction to the drag formula of Stokes, which holds when $T = 0$, was obtained. Maxworthy (1965) measured experimentally the drag on a sphere as it moves along the axis of a rotating viscous fluid at small values of R and T. He found that for a fixed value of R the drag is increased as the rotation parameter T increases. His results in general confirm the theory of Childress. However, this theory is valid only for such small values of R and T that more comprehensive solutions of the Navier–Stokes equations are necessary to confirm Maxworthy's experiments even over this range of R and T; such solutions are sought numerically in the present study.

In a more extensive set of experiments for a larger range of values of R and T, Maxworthy (1970) observed forward separation of the axisymmetric flow past a sphere moving in a rotating fluid with the formation of an upstream separation bubble. The sphere is surrounded by a thin annular region within which the velocity is larger than the mean velocity of the approaching flow and a vortex-jump phenomenon is found to occur in several regions. The boundary-layer separation from a sphere in a rotating fluid has been calculated by Miles (1971) using a least-squares approximation and on the hypothesis that the flow exerts no upstream influence. A reversed flow is found to occur in the neighbourhood of the forward stagnation point for $1/R_0 > 2{\cdot}2$

and is accompanied by a forward separation bubble such as observed by Maxworthy. For small values of R and T, a perturbation solution giving the flow past a stationary and spinning sphere in a rotating fluid has been obtained by Singh (1975*a*, *b*). A region of reversed flow and vortex formation is found to occur near the front or rear stagnation point, or both, depending upon the values of R and T and the angular velocity of the sphere.

Hocking *et al.* (1979) calculated the drag on a sphere moving along the axis of a long finite rotating container when the length of the Taylor column is comparable to the axial length of the container. The Rossby and Ekman numbers were both assumed to be small. The determination of the drag involves solving dual integral equations. The drag on the sphere is found to be greater in the case when the sphere is in the container than its value in an unbounded fluid, but the increased value is smaller than that measured by Maxworthy (1970). Recently, Dennis & Ingham (1981) have described a specialized numerical method for determining the flow created by the slow motion of a sphere in a viscous rotating fluid. The method is based on finite-difference equations approximating the basic governing equations which, because of their specialized nature, involve the exponential function. By expanding the exponentials in powers of their exponents an approximation is arrived at which is very suitable in the numerical calculations. Some illustrations of the method were given which confirm the theoretical work of Childress (1964) for very low values of R in the case $T = 0$ but these illustrations did not cover the range of the experimental work of Maxworthy (1965).

In the present paper the object is to obtain numerical solutions of the equations governing the problem considered experimentally by Maxworthy (1965) over the same range of small values of R and T. Two independent numerical schemes are employed and the results are used to check each other. The first scheme is the two-dimensional finite-difference method described in detail by Dennis & Ingham (1981) and the second is the method of series truncation. Some details of recent applications of this latter method to determine flows due to rotating spheres without translation have been given by Dennis & Singh (1978) and Dennis, Singh & Ingham (1980) in the case of steady-state flows and by Dennis & Ingham (1979) for an unsteady flow. In the series-truncation method the basic partial differential equations are reduced to infinite sets of ordinary differential equations by means of a series substitution. For the present problem the independent variables are spherical polar co-ordinates (r, θ, ϕ) with all quantities independent of ϕ owing to axial symmetry. The modified co-ordinate $\xi = \ln(r/a)$ is introduced and then all quantities depend only on (ξ, θ). Finally, the dependent variables are expressed as series of orthogonal Gegenbauer functions with argument $\mu = \cos\theta$ and variable coefficients which are functions of ξ. On substitution of the series in the Navier–Stokes equations, the problem is reduced to the solution of three infinite sets of second-order ordinary differential equations in three dependent variables.

The ordinary differential equations are reduced to a finite set by truncation of the series to a finite number of terms and the finite set of equations is solved by numerical methods. The numerical methods are in principle similar to those employed by Dennis & Singh (1978) and Dennis *et al.* (1980) except that, whereas in both of those cases the integration range was finite, here it is infinite. In theory, boundary conditions are specified at $\xi = 0$ and as $\xi \to \infty$, but in practice the conditions as $\xi \to \infty$

must be replaced by conditions at some large distance $\xi = \xi_\infty$. This problem is dealt with by performing a solution with some definite specified value of ξ_∞ at which the conditions as $\xi \to \infty$ are assumed and then, after this approximate solution has been obtained, increasing ξ_∞ and obtaining a new approximate solution. The process is then repeated until there is no significant change in the main properties of the solution to within a definite tolerance, taken to be about 0·1 %. Numerical solutions obtained by means of this procedure using finite-difference approximations, with grid sizes judged to be sufficiently small and a sufficiently large number of terms in the approximating series expansions, are found to give results in good agreement with the experiments of Maxworthy (1965) and to be consistent as $R \to 0$ with the theory of Childress (1964).

2. Formulation of the problem

We take co-ordinate axes which are fixed in direction and with origin at the centre of a sphere of radius a which is moving with constant velocity U in the negative z-direction in an incompressible viscous fluid. The whole mass of fluid is rotating with axial symmetry about the z-axis and in such a way that the angular velocity at large enough distances from the sphere is constant and equal to ω_0. The sphere is assumed to be free to rotate about the z-axis in such a way that the torque exerted on it by the rotating fluid is zero. We use spherical polar co-ordinates (r, θ, ϕ) and since the motion is axially symmetric about the z-axis, all quantities are independent of ϕ. If the transformation $\xi = \ln(r/a)$ is used, the Navier–Stokes equations can be expressed in the form (Dennis & Ingham 1981) given by

$$D^2\Omega = \frac{Re^{-\xi}}{\sin\theta}\left(\frac{\partial\psi}{\partial\theta}\frac{\partial\Omega}{\partial\xi} - \frac{\partial\psi}{\partial\xi}\frac{\partial\Omega}{\partial\theta}\right), \tag{1}$$

$$D^2\psi = -e^{2\xi}\zeta, \tag{2}$$

$$D^2\zeta = \frac{Re^{-\xi}}{\sin\theta}\left[\left(\frac{\partial\psi}{\partial\theta}\frac{\partial\zeta}{\partial\xi} - \frac{\partial\psi}{\partial\xi}\frac{\partial\zeta}{\partial\theta}\right) + 2\left(\cot\theta\frac{\partial\psi}{\partial\xi} - \frac{\partial\psi}{\partial\theta}\right)\zeta - 2R^{-2}T^2\left(\cot\theta\frac{\partial\Omega}{\partial\xi} - \frac{\partial\Omega}{\partial\theta}\right)\Omega\right]. \tag{3}$$

In these equations

$$D^2 = \frac{\partial^2}{\partial\xi^2} - \frac{\partial}{\partial\xi} + \sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right),$$

and ψ, $\Omega e^{-\xi}/\sin\theta$, $\zeta e^{-\xi}/\sin\theta$ are the dimensionless stream function, angular velocity and vorticity respectively. The dimensionless velocity components (v_r, v_θ, v_ϕ), obtained by dividing the dimensional components by U, are related to ψ and Ω by the equations

$$v_r = \frac{e^{-2\xi}}{\sin\theta}\frac{\partial\psi}{\partial\theta}, \quad v_\theta = -\frac{e^{-2\xi}}{\sin\theta}\frac{\partial\psi}{\partial\xi}, \quad v_\phi = \frac{\Omega e^{-\xi}}{\sin\theta}. \tag{4}$$

The actual dimensional variables, denoted by quantities with asterisks, are related by the equations

$$v_r^* = \frac{1}{r^2\sin\theta}\frac{\partial\psi^*}{\partial\theta}, \quad v_\theta^* = -\frac{1}{r\sin\theta}\frac{\partial\psi^*}{\partial r}, \quad v_\phi^* = \frac{\Omega^*}{r\sin\theta}; \tag{5}$$

and hence we have the relations

$$\psi^* = a^2 U\psi, \quad \Omega^* = a^2\omega_0\Omega, \tag{6}$$

between dimensional and dimensionless quantities.

It is assumed that during the course of the motion the sphere is free to rotate with a constant angular velocity subject to no resultant torque exerted on it by the fluid. The boundary conditions on the dimensionless functions ψ and Ω are then

$$\left.\begin{aligned} &\psi = \partial\psi/\partial\xi = 0, \quad \Omega = \tilde{\omega}\sin^2\theta \quad \text{when} \quad \xi = 0; \\ &\psi \sim \tfrac{1}{2}e^{2\xi}\sin^2\theta, \quad \Omega \sim e^{2\xi}\sin^2\theta, \quad \zeta\to 0 \quad \text{as} \quad \xi\to\infty \end{aligned}\right\} \quad (0 \leqslant \theta \leqslant \pi), \tag{7}$$

where $\tilde{\omega}$ is the ratio of the angular velocity of the sphere to that of the fluid at large distances from the sphere. These conditions are appropriate for the functions governed by (1)–(3), but it is also possible to consider the problem in terms of a perturbation from the flow at large distances from the sphere by substituting

$$\psi = \tfrac{1}{2}e^{2\xi}\sin^2\theta + \tilde{\psi}, \quad \Omega = e^{2\xi}\sin^2\theta + \tilde{\Omega}. \tag{8}$$

The equations satisfied by $\tilde{\Omega}$, $\tilde{\psi}$ and ζ are easily found from (1)–(3) and will not be given in detail. The boundary conditions at the surface of the sphere and at large distances are

$$\left.\begin{aligned} &\tilde{\psi} = -\tfrac{1}{2}\sin^2\theta, \quad \partial\tilde{\psi}/\partial\xi = -\sin^2\theta, \quad \tilde{\Omega} = (\tilde{\omega}-1)\sin^2\theta \quad \text{when} \quad \xi = 0; \\ &e^{-2\xi}\partial\tilde{\psi}/\partial\theta\to 0, \quad e^{-2\xi}\partial\tilde{\psi}/\partial\xi\to 0, \quad \Omega\to 0 \quad \text{as} \quad \xi\to\infty. \end{aligned}\right\} \tag{9}$$

The conditions at large distances from the sphere are that the velocity components of the perturbation must tend to zero. We shall return to these conditions later.

Two methods of solution have been adopted, both involving numerical methods. In one of these we work in terms of $\tilde{\Omega}$, $\tilde{\psi}$ and ζ and use the method of series truncation, in which the governing equations are reduced to infinite sets of ordinary differential equations. In the second (1)–(3) are approximated by a two-dimensional finite-difference method subject to the boundary conditions (7), where the conditions as $\xi\to\infty$ are assumed to hold at some large-enough finite value $\xi = \xi_\infty$. The essential details of this method have been given by Dennis & Ingham (1981) and we shall mainly give details of the series-truncation method here.

3. Solution by the series-truncation method

The application of this method to the present problem follows more or less the work of Dennis & Singh (1978) but, since in the present case we use the perturbation functions defined in (8), extra terms appear in the basic equations. The equations governing $\tilde{\Omega}$, $\tilde{\psi}$ and ζ are, in fact, the same as (1)–(3) with the interchange of Ω and ψ with the corresponding perturbation functions and the addition of the respective terms

$$Re^{\xi}\left\{\left(\cos\theta\frac{\partial\tilde{\Omega}}{\partial\xi} - \sin\theta\frac{\partial\tilde{\Omega}}{\partial\theta}\right) - 2\left(\cos\theta\frac{\partial\tilde{\psi}}{\partial\xi} - \sin\theta\frac{\partial\tilde{\psi}}{\partial\theta}\right)\right\}, \tag{10}$$

$$Re^{\xi}\left\{\left(\cos\theta\frac{\partial\zeta}{\partial\xi} - \sin\theta\frac{\partial\zeta}{\partial\theta}\right) - 2R^{-2}T^2\left(\cos\theta\frac{\partial\tilde{\Omega}}{\partial\xi} - \sin\theta\frac{\partial\tilde{\Omega}}{\partial\theta}\right)\right\} \tag{11}$$

to the right-hand sides of (1) and (3). The assumption of expansions for the dependent variables in series of Gegenbauer functions is similar to that of Dennis & Singh (1978) and Dennis *et al.* (1980) except that here there is no symmetry of the flow about $\theta = \frac{1}{2}\pi$ and the assumptions must reflect this fact. We thus assume

$$\tilde{\Omega} = \sum_{n=1}^{\infty} I_{n+1}(\mu) f_n(\xi), \quad \tilde{\psi} = \sum_{n=1}^{\infty} I_{n+1}(\mu)\, g_n(\xi), \quad \zeta = \sum_{n=1}^{\infty} I_{n+1}(\mu)\, h_n(\xi). \tag{12}$$

Here $\mu = \cos\theta$ and $I_n(\mu)$ are the Gegenbauer functions of the first kind and of order n. Sampson (1891) discussed the properties of the Gegenbauer functions, which form a set of orthogonal functions in the interval $\mu = -1$ to $\mu = 1$; and the main properties required in the present application are given by Dennis & Singh (1978).

The governing equations for $\tilde{\Omega}$, $\tilde{\psi}$ and ζ are first expressed in terms of μ and then, on substitution of the expansions (12), we obtain

$$f_n'' - f_n' - n(n+1) f_n = n(n+1)(2n+1)\, Re^{\xi} R_n, \tag{13}$$

$$g_n'' - g_n' - n(n+1)\, g_n = -e^{2\xi} h_n, \tag{14}$$

$$h_n'' - h_n' - n(n+1)\, h_n = n(n+1)(2n+1)\, Re^{\xi} S_n, \tag{15}$$

where the primes denote differentiation with respect to ξ. The right-hand sides of (13) and (15) involve summations R_n and S_n coming from the right-hand sides of (1) and (3), when expressed in terms of $\tilde{\Omega}$ and $\tilde{\psi}$. These may be written as

$$\begin{aligned} R_n = {} & e^{-2\xi} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} [L(m,n,l) f_l g_m' - L(l,n,m) f_l' g_m] \\ & - \sum_{m=1}^{\infty} [L(m,n,1)(f_m' - 2g_m') - 2L(n,1,m)(f_m - 2g_m)], \end{aligned} \tag{16}$$

$$\begin{aligned} S_n = {} & e^{-2\xi} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} [\{L(n,l,m) + M(m,l,n)\} g_l' h_m - L(m,n,l)(h_m' - 2h_m) g_l \\ & - R^{-2}T^2\{M(l,m,n) f_l' + 2L(n,m,l) f_l\} f_m] \\ & - \sum_{m=1}^{\infty} [L(m,n,1)(h_m' - 2R^{-2}T^2 f_m') - 2L(n,1,m)(h_m - 2R^{-2}T^2 f_m)]. \end{aligned} \tag{17}$$

The quantities $L(l,m,n)$ and $M(l,m,n)$ are exactly the integrals involving products of Gegenbauer and derived functions defined by Dennis & Singh (1978). They can be expressed in terms of the Wigner 3-j symbols in the form

$$L(l,m,n) = [l(l+1)\, m(m+1)]^{-\frac{1}{2}} \begin{pmatrix} l & m & n \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} l & m & n \\ 0 & 0 & 0 \end{pmatrix}, \tag{18}$$

$$M(l,m,n) = \left[\frac{(l-1)(l+2)}{lmn(l+1)(m+1)(n+1)}\right]^{\frac{1}{2}} \begin{pmatrix} l & m & n \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} l & m & n \\ 0 & 0 & 0 \end{pmatrix} - L(l,m,n). \tag{19}$$

The Wigner 3-j symbols have been discussed fully by Rotenberg *et al.* (1959) and by Talman (1968). Those required in (18) and (19) were calculated by Dennis & Singh (1978) from their definitions by a computer algorithm.

Boundary conditions for the sets of equations (13), (14) and (15) are obtained by considering the series (12) in relation to the boundary conditions for the basic functions $\tilde{\Omega}$, $\tilde{\psi}$ and ζ. The conditions when $\xi = 0$ may be deduced from (9), although the quantity $\tilde{\omega}$ is not known and must be determined as part of the solution. The conditions for the functions $f_n(\xi)$ are deduced from the assumption that the torque exerted by the fluid on the sphere is zero. If T^* is the torque, Dennis & Singh (1978) give

$$T^* = 2\pi\rho\nu a^3\omega_0 \int_0^\pi \left(\frac{\partial\Omega}{\partial\xi} - 2\Omega\right)_{\xi=0} \sin\theta\, d\theta, \tag{20}$$

where ρ is the density of the fluid. If we put $T^* = 0$ and substitute in the integral using (8) and (9), then

$$f_1'(0) = 2f_1(0), \quad f_n(0) = 0 \quad (n = 2, 3, 4, \ldots). \tag{21}$$

The conditions for $g_n(\xi)$ when $\xi = 0$ are found by substituting the series for $\tilde{\psi}$ given in (12) into the conditions on this function and $\partial\tilde{\psi}/\partial\xi$ in (9), which yields

$$g_1(0) = -1, \quad g_1'(0) = -2, \quad g_n(0) = g_n'(0) = 0 \quad (n = 2, 3, 4, \ldots). \tag{22}$$

There are no direct conditions for $h_n(\xi)$ when $\xi = 0$, although conditions are needed in order to obtain numerical solutions of (15). The values $h_n(0)$ are determined as part of the solution in a manner similar to that used by Dennis & Singh (1978).

The conditions to be satisfied for large ξ, which follow from (9), are that

$$f_n(\xi) \to 0, \quad e^{-2\xi} g_n(\xi) \to 0, \quad e^{-2\xi} g_n'(\xi) \to 0 \quad \text{as} \quad \xi \to \infty. \tag{23}$$

In practice they must be satisfied approximately by imposing conditions that are effectively equivalent, to some measure of approximation, at a finite value $\xi = \xi_\infty$. The satisfaction of these conditions follows quite closely the method of Dennis & Singh (1978). In practice, instead of working in terms of $g_n(\xi)$ we employ the functions $G_n(\xi)$ defined by the transformation

$$g_n(\xi) = e^{\frac{1}{2}\xi} G_n(\xi) \quad (n = 1, 2, \ldots). \tag{24}$$

Substitution into (14) gives the set of equations

$$G_n'' - k^2 G_n = r_n, \tag{25}$$

where

$$r_n(\xi) = -e^{-\frac{3}{2}\xi} h_n(\xi), \quad k = n + \tfrac{1}{2}. \tag{26}$$

The boundary conditions for (25) are

$$G_1(0) = -1, \quad G_1'(0) = -\tfrac{3}{2}, \quad G_n(0) = G_n'(0) = 0 \quad (n = 2, 3, 4, \ldots), \tag{27}$$

$$e^{-\frac{3}{2}\xi} G_n(\xi) \to 0, \quad e^{-\frac{3}{2}\xi} G_n'(\xi) \to 0 \quad \text{as} \quad \xi \to \infty. \tag{28}$$

In the problem of flow between two rotating spheres considered by Dennis & Singh (1978), the functions $G_n(\xi)$ and $G_n'(\xi)$ must vanish on each sphere. In the present problem, if the conditions in (28) are to be approximated at a finite value $\xi = \xi_\infty$, we must either assume that $G_n(\xi)$ and $G_n'(\xi)$ vanish at $\xi = \xi_\infty$ or make use of an asymptotic expression for $G_n(\xi)$ valid as $\xi \to \infty$. It is difficult to find such an expression in closed form and thus we assume as conditions at large distances that

$$f_n(\xi_\infty) = 0, \quad G_n(\xi_\infty) = 0, \quad G_n'(\xi_\infty) = 0, \tag{29}$$

and subsequently ensure in the numerical process that ξ_∞ has been taken large enough for the approximation to be valid. In effect we are replacing (28) by equivalent conditions on a solid sphere at $\xi = \xi_\infty$. We shall return to this point when the numerical results are described.

We may now follow the procedure outlined by Dennis & Singh (1978). If (25) is multiplied by $e^{\pm k\xi}$ for a given value of n and both sides integrated with respect to ξ from $\xi = 0$ to $\xi = \xi_\infty$ we obtain

$$\int_0^{\xi_\infty} e^{\pm k\xi} r_n(\xi)\, d\xi = \left.\begin{matrix} 0 \\ 3\delta_{1,n} \end{matrix}\right\}, \tag{30}$$

where $\delta_{1,n}$ is the Kronecker delta and the upper and lower values on the right-hand side correspond to the upper and lower signs of the exponent on the left. The two equations (30) are used to obtain estimates of $r_n(0)$ and $r_n(\xi_\infty)$ by expressing the appropriate integral as a quadrature formula. The method is almost identical to that of Dennis & Singh (1978) and need only be briefly summarized, using similar notation. The first integral in (30) with the positive sign taken in the exponent in the integrand is written approximately as a quadrature formula

$$c_0 r_n(\xi_\infty) + c_1 r_n(\xi_\infty - h) + \ldots + c_p r_n(0) = 0, \tag{31}$$

where the integrand has been divided into p equal intervals of length h. This can be expressed as

$$c_0 r_n(\xi_\infty) + c_p r_n(0) + Q = 0, \tag{32}$$

where Q is the sum over internal values. The second integral in (30) can similarly be expressed as

$$c_0' r_n(\xi_\infty) + c_p' r_n(0) + Q' = 0. \tag{33}$$

In this case Q' consists of a weighted sum of values of $r_n(\xi)$ and in addition includes a contribution involving $\delta_{1,n}$ when $n = 1$.

The quantities Q and Q' in (32) and (33) are known approximately during the course of an iterative procedure of obtaining a numerical solution and hence estimates of $r_n(0)$ and $r_n(\xi_\infty)$ may be found by solving (32) and (33). In this way, through (26), estimates α_n and β_n of $h_n(\xi)$ at $\xi = 0$ and ξ_∞ are obtained to use as boundary conditions for the set of equations (15) in the form

$$h_n(0) = \alpha_n, \quad h_n(\xi_\infty) = \beta_n \quad (n = 1, 2, \ldots). \tag{34}$$

The quadrature formula employed for approximating the integral in (30) is a specialized one. The total number of intervals p is even and over each successive pair of intervals a modified type of Simpson's formula is used in which $r_n(\xi)$ is approximated by a parabola rather than the whole integrand. The details are given by Dennis & Singh (1978). In the present case if, for a given value of h, $\kappa(k)$ is defined by

$$\kappa(k) = \frac{1}{k} - \frac{1}{2hk^2}(3 + e^{-2kh}) + \frac{1}{h^2k^3}(1 - e^{-2kh}), \tag{35}$$

where k is given in (26), then the coefficients required in (32) and (33) are given by

$$e^{-k\xi_\infty} c_0 = c_p' = \kappa(k), \quad e^{k\xi_\infty} c_0' = c_p = \kappa(-k). \tag{36}$$

The set of equations (14) is integrated by step-by-step methods by substituting

$$u_n = G_n' - kG_n, \quad v_n = G_n' + kG_n. \tag{37}$$

The functions u_n and v_n satisfy

$$u_n' + ku_n = r_n, \quad v_n' - kv_n = r_n, \tag{38}$$

subject to the conditions

$$u_n(0) = 0, \quad u_n(\xi_\infty) = 0, \quad v_n(0) = -3\delta_{1,n}, \quad v_n(\xi_\infty) = 0. \tag{39}$$

The equations (38) are exactly the equations solved by Dennis & Singh (1978), although the boundary conditions for $v_n(\xi)$ are slightly different, and the same specialized integration techniques are applicable. For a given integer n, the first of (38) is integrated in the direction of increasing ξ subject to $u_n(0) = 0$, since the integration is stable in that direction. The second of (38) is then integrated subject to $v_n(\xi_\infty) = 0$ in the decreasing direction of ξ, since that integration is stable. Provided that the conditions (30) have been satisfied properly, all four conditions in (39) are found to be satisfied by this procedure. All the necessary formulae are given by Dennis & Singh (1978).

The two sets of equations (13) and (15) are solved using standard finite-difference methods. A typical equation of either set can be written

$$y_n''(\xi) + a_n(\xi)\, y_n'(\xi) + b_n(\xi)\, y_n(\xi) = s_n(\xi), \tag{40}$$

where $a_n(\xi)$ and $b_n(\xi)$ will include terms associated with the particular $y_n(\xi)$ or its derivative under consideration coming from the nonlinear sums $R_n(\xi)$ or $S_n(\xi)$ in (13) or (15). Equation (40) is approximated using central differences, which gives

$$\{1 - \tfrac{1}{2}ha_n(\xi)\}\, y_n(\xi - h) - \{2 - h^2 b_n(\xi)\}\, y_n(\xi) + \{1 + \tfrac{1}{2}ha_n(\xi)\}\, y_n(\xi + h) - h^2 s_n(\xi) = 0, \tag{41}$$

at each station ξ of the same grid structure used to solve the set of equations (14). There is a set of finite-difference equations of type (41) for each differential equation. The set is solved iteratively by the Gauss–Seidel procedure subject to two-point boundary conditions. For (13) these are (21) together with the first condition of (29); for (15) they are the conditions (34) that have been deduced during the solution procedure of (14). The whole set of procedures of solving (13), (14), (15) and calculating α_n and β_n in (34) is performed in an iterative loop which is repeated until convergence. The boundary condition for $f_1(\xi)$ at $\xi = 0$ in (21) is satisfied by expressing $f_1'(0)$ in central differences, which gives the approximation

$$f_1(-h) = 2(1 + h^2) f_1(0) - f_1(h). \tag{42}$$

This may be used to eliminate the external value which appears when (41) is applied at $\xi = 0$ taking $y_n(\xi)$ to be $f_1(\xi)$.

4. Finite-difference method and solution procedures

Several different two-dimensional finite-difference schemes can be used to approximate (1), (2) and (3) subject to the boundary conditions given by (7), and these vary considerably in accuracy and efficiency. Central differences can be used, but difficulty may then be encountered in solving the finite-difference equations by iterative techniques, which may fail to converge unless severe under-relaxation is used, particularly for high Reynolds numbers. The use of upwind and downwind differences in approximating the Navier–Stokes equations has been studied by Greenspan (1968*a*), Gosman *et al.* (1969), and Runchal, Spalding & Wolfshtein (1969). Approximation by such methods improves the convergence of iterative procedures of solution but at the expense of being only first-order accurate. The improvement in convergence arises from the fact that the finite-difference equations are associated with matrices which are diagonally dominant. Allen & Southwell (1955), Allen (1962), Dennis (1960, 1973), Spalding (1972) and Roscoe (1975, 1976) have developed methods that are second-order accurate and have associated matrices which are diagonally dominant, but the coefficients of the finite-difference equations involve exponentials.

A recent paper by Dennis & Hudson (1978) has shown that, for the vorticity-transport equation of the two-dimensional Navier–Stokes equations in Cartesian co-ordinates, a simple expansion procedure applied to the approximation of Dennis (1960) gives a set of finite-difference equations with coefficients free from exponentials but which preserves second-order accuracy and also has an associated matrix which is diagonally dominant. Dennis, Ingham & Cook (1979) extended the method to three-dimensional flows and subsequently Dennis & Ingham (1981) applied similar methods to the equations (1)–(3) governing the present problem. The approximation to (2) is the standard central-difference one but to (1) and (3) it is not. The approximations to both (1) and (2) have associated matrices that are diagonally dominant but it is not possible to demonstrate a similar property for the approximation to (3). However, the method deals in a more satisfactory way with the nonlinear terms in (1) and (3) than the central-difference method and as a consequence may be expected to lead to improved convergence of iterative methods of solution. This was the case in trial calculations carried out by Dennis & Ingham (1981). Similar results have been found in the present work; no under-relaxation was necessary in the iterative cycles.

The finite-difference equations employed to approximate (1)–(3) are given by Dennis & Ingham (1981) and need not be given again here. The solution procedures used here are identical. The boundary conditions for ψ, Ω and ζ as $\xi \to \infty$ given in (7) must be satisfied at a large-enough value $\xi = \xi_\infty$ and then the effect of increasing ξ_∞ is studied. Some results of making tests of this nature were given by Dennis & Ingham (1981) and some further results are given in §5 of the present paper. The initial set of approximations to the solutions of (1)–(3) needed to start the iterative process of solution for small values of R and T is obtained from the results for $T = 0$ computed by Dennis & Ingham (1981).

In the case of solutions computed using the series-truncation method, the very small value $\xi_\infty = \ln 2$ was first used for the outer boundary in each case, corresponding to an outer sphere of radius $r/a = 2$. Solutions were first obtained using only one term in each of the series (12) and all others were assumed to be zero; this is the first truncation, $n_0 = 1$. For this order of truncation at small values of R and T, an initial approximation was obtained by putting $R = T = 0$ in (13)–(15) and solving these equations for $n = 1$ subject to the corresponding boundary conditions obtained from (21), (22), (29) and (30). With $\xi_\infty = \ln 2$ this gives

$$\left.\begin{aligned} f_1^{(0)}(\xi) &= 0, \quad g_1^{(0)}(\xi) = \tfrac{1}{17}(-9e^{4\xi}+122e^{2\xi}-186e^{\xi}+56e^{-\xi}),\\ h_1^{(0)}(\xi) &= \tfrac{1}{17}(90e^{2\xi}-376e^{-\xi}), \end{aligned}\right\} \tag{43}$$

where the superscript indicates the order of the iterate, in this case zero. This initial approximation determines approximations to $R_1(\xi)$ and $S_1(\xi)$ in (13) and (15) and then these two equations are solved approximately by applying the Gauss–Seidel procedure to the finite-difference equations (41) subject to their boundary conditions. In the case of $h_n(\xi)$ the conditions (34) are used with $\alpha_1^{(0)}$ and $\beta_1^{(0)}$ determined from the approximation $h_1^{(0)}(\xi)$ in (43).

Once a new approximation to $h_1(\xi)$ has been found, new approximations to α_1 and β_1 can be found by methods described in § 3, and then (25) is integrated for $n = 1$ to determine $G_1(\xi)$ and $G_1'(\xi)$ approximately. This completes one major iterative cycle. This is then repeated until the various functions converge to limits. Truncations of higher order for the same values of R and T are obtained by using lower truncations as starting approximations, assuming all new functions introduced to be initially zero. The only new point for a truncation of order n_0 is that each step of the iterative cycle is carried out for all equations of the set from $n = 1$ to n_0 in turn. The boundary values of $h_n(\xi)$ at $\xi = 0$ and $\xi = \xi_\infty$ are calculated using the smoothing process

$$h_n^{(m+1)}(\xi) = \lambda\tilde{h}_n(\xi) + (1-\lambda)\,h_n^{(m)}(\xi) \quad (\xi = 0, \xi_\infty). \tag{44}$$

Here the superscript m refers to successive iterates, the values $\tilde{h}_n(\xi)$ for $\xi = 0$, ξ_∞ are values calculated using (26) from the values $r_n(0)$ and $r_n(\xi_\infty)$ determined from (32) and (33), and λ is a relaxation parameter in the range $0 < \lambda \leqslant 1$. It would be possible to associate distinct values of λ with the individual cases $\xi = 0$, $\xi = \xi_\infty$ in (44) but this was not done.

For given values of R and T the iterative procedure of solution of (13)–(15) for a given truncation n_0 is repeated until convergence. This is decided by the test

$$|h_n^{(m+1)}(\xi) - h_n^{(m)}(\xi)| < 10^{-5} \quad (0 < \xi < \xi_\infty, \quad n = 1, 2, \ldots, n_0). \tag{45}$$

It is only necessary to test the one set of functions $h_n(\xi)$, because, when these have converged to limit solutions, the other sets of functions are found by inspection to have converged to limit solutions also. After the solution for $n_0 = 3$ had been obtained using these procedures, the effect of increasing ξ_∞ was studied. This parameter was increased to $\xi_\infty = 3$, 3·5, 4 and 4·5 successively, and approximate solutions obtained for each case. Finally, in obtaining solutions for higher values of R and T, the approximations already obtained for lower values of R and T were used as starting assumptions. By this means solutions were obtained for $R = 0{\cdot}05$, 0·1, 0·2 and 0·5, and for each value of R the values $T = 0{\cdot}05$, 0·12 and 0·25 of the Taylor number were considered. The maximum value $n_0 = 4$ was used in each case, which appeared to give a satisfactory enough approximation. The values of λ used in (44) were as follows. For all of $R = 0{\cdot}05$, 0·1 and 0·2 the value $\lambda = 0{\cdot}05$ was used for $T = 0{\cdot}05$, 0·12 and $\lambda = 0{\cdot}02$ was used for $T = 0{\cdot}25$. For $R = 0{\cdot}5$ the only change was to use $\lambda = 0{\cdot}04$ for $T = 0{\cdot}05$, 0·12.

5. Results

Numerical calculations were performed using both the two-dimensional specialized finite-difference scheme and the series-truncation method. In the two-dimensional scheme most calculations were carried out with grid sizes of $\frac{1}{30}\pi$ and $\frac{1}{60}\pi$ in the angular direction and the results with these two grid sizes were virtually the same. In the radial direction a grid size as small as 0·01 was used, but in general a grid size of 0·05

ξ_∞	$T = 0{\cdot}025$ Series truncation	Finite-difference equations	$T = 0{\cdot}12$	$T = 0{\cdot}25$	$T = 0{\cdot}5$
2·0	—	1·316	1·340	1·412	1·617
2·5	—	1·204	1·282	1·400	1·626
3·0	1·142	1·152	1·261	1·398	1·632
3·5	1·111	1·125	1·254	1·398	1·630
4·0	1·107	1·112	1·252	1·398	1·631
4·5	1·108	1·107	1·251	—	—
5·0	—	—	1·251	—	—

TABLE 1. Variation of D/D_s for $R = 0{\cdot}12$ with the position of the outer boundary conditions at various values of T

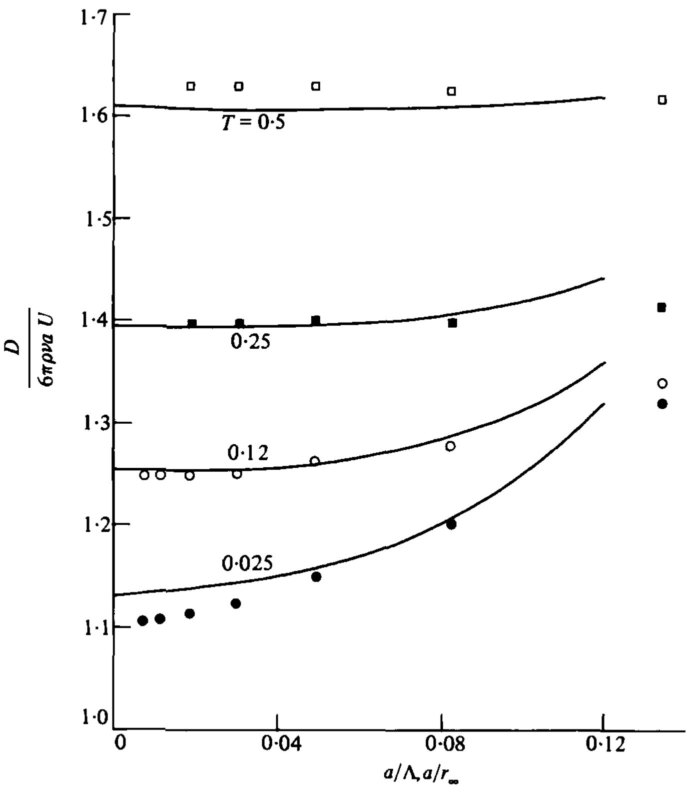

FIGURE 1. Values of the drag on the sphere for $R = 0{\cdot}12$. ——, experimental results of Maxworthy (1965) versus a/Λ. Present numerical calculations versus a/r_∞: ●, $T = 0{\cdot}025$; ○, 0·12; ■, 0·25; □, 0·5.

was more than adequate in most calculations in order to obtain accurate results. It was possible to use the Gauss–Seidel iterative method in all the iterative procedures without any under-relaxation and, moreover, the finite-difference formula used to calculate the vorticity on the sphere (Dennis & Ingham 1981) could be employed without under-relaxation. It was found, however, that the position of the boundary $\xi = \xi_\infty$ was critical in all cases.

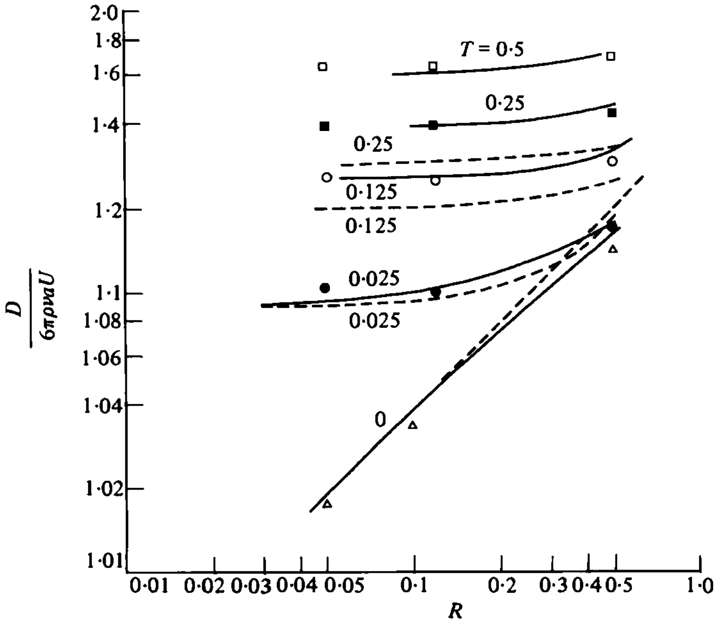

FIGURE 2. Variation of the drag on the sphere with Reynolds number. - - - - -, theoretical results of Childress (1964); ——, experimental results of Maxworthy (1965). Present calculations: △, $T = 0$; ●, 0·025; ○, 0·12; ■, 0·25; □, 0·5.

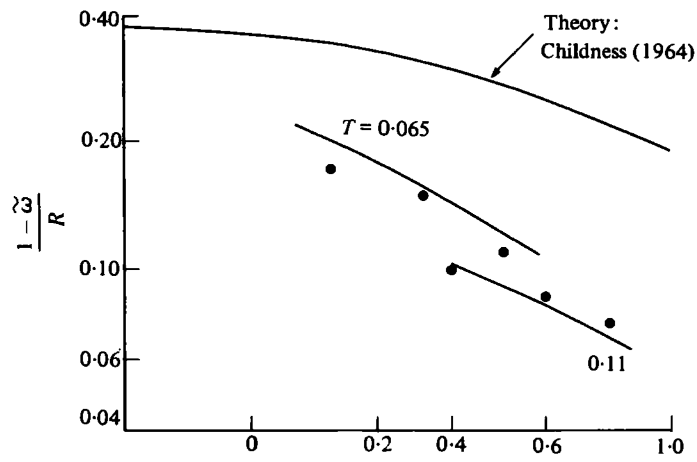

FIGURE 3. Variation of the differential rotation of the sphere with $\alpha = 2T/R^2$: ——, experimental results of Maxworthy (1965); ●, present calculations.

The drag D on the sphere is given by the formula

$$D = -\tfrac{1}{3}D_s \int_0^{\pi} (pR\cos\theta + \zeta)\sin\theta\, d\theta, \tag{46}$$

where D_s is the drag for Stokes flow, given by $D_s = 6\pi\rho\nu Ua$. Here p is the dimensionless pressure in the fluid and ρ is the density. Evaluation of the drag from (46) was

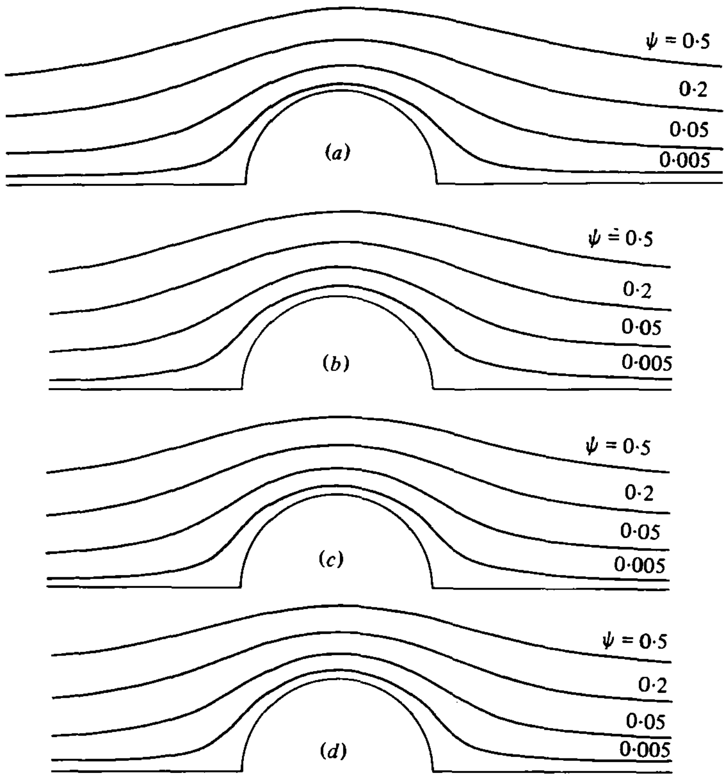

FIGURE 4. Streamlines for $R = 0{\cdot}5$: (a) $T = 0$; (b) 0·12; (c) 0·25; (d) 0·5.

performed using the results obtained from both numerical techniques; for the series-truncation method (46) becomes

$$D/D_{\mathrm{S}} = \tfrac{1}{9}\{2h_1(0) - h_1'(0)\}. \tag{47}$$

Table 1 shows typical results for D/D_{S} as a function of ξ_∞ for $R = 0{\cdot}12$. In the first instance, a comparison between the results for the two independent methods for $T = 0{\cdot}025$ is given. There is reasonable consistency between the two sets of results; this was also found to be the case for other values of R and T. Thus the values for $T = 0{\cdot}12$, 0·25 and 0·5 presented in table 1 are those which were considered to be the most accurate after carrying out different solutions for different grid sizes using both methods, and for different orders of truncation when using the series-truncation method. It may be noticed from table 1 that a smaller value of ξ_∞ is needed to obtain satisfactory results as T increases.

Maxworthy (1965) noted in his experimental work that the ratio of the radius of the sphere to the radius of the cylinder in which the experiments were carried out, denoted by Λ, had a very significant influence on the measured drag, especially for small T. Thus it is not surprising that, as indicated in table 1, the value of ξ_∞ has to be large to deal with the boundary condition at infinity adequately. Figure 1 shows the experimental variation of D/D_{S} as a function of a/Λ obtained by Maxworthy for $R = 0{\cdot}12$ and $T = 0{\cdot}025$, 0·12, 0·25 and 0·5. Also shown on the same figure is the variation of D/D_{S} as a function of a/r_∞ obtained from the present results, where $r_\infty = a\,e^{\xi_\infty}$, for the same values of R and T. As one would expect, the effect of imposing

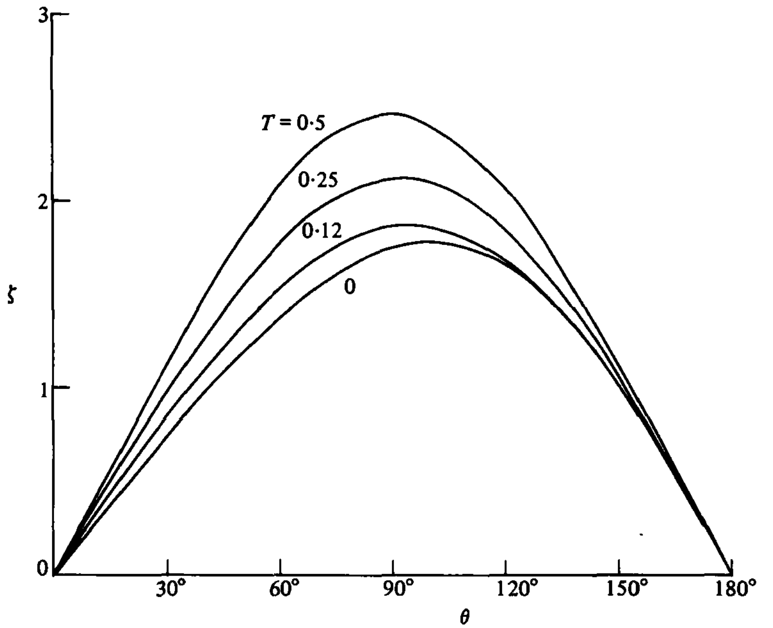

FIGURE 5. Surface vorticity on the sphere for $R = 0{\cdot}5$ and various values of T.

numerically the outer boundary condition at a finite distance has many similarities with the positioning of the outer boundary in the experimental results. In all the calculations, results were obtained for several values of ξ_∞; these were extrapolated to estimate the results in an unbounded fluid.

Figure 2 shows the calculated variation of D/D_s as a function of R at various values of T, together with the experimental results of Maxworthy (1965) and the theoretical results of Childress (1964). This latter theory is only valid for small values of both R and T. It is seen that all three sets of results are reasonably consistent. Figure 3 shows the variation of the differential rotation $\hat{\omega} = 1 - \bar{\omega}$ as a function of the parameter $\alpha = 2T/R^2$ according to the present results and the work of Childress and Maxworthy. There is again very good agreement between the present results and the experimental measurements. Streamlines for $R = 0{\cdot}5$ and $T = 0$, $0{\cdot}12$, $0{\cdot}25$ and $0{\cdot}5$ are shown in figure 4 and the surface vorticity on the sphere for the same values of R and T in figure 5. For the case $R = 0{\cdot}5$, $T = 0$ these results agree with results previously presented, e.g. by Dennis & Walker (1971). The effect of increasing T for a given value of R is to make the flow more symmetrical and also to increase the vorticity on the surface of the sphere, which tends to increase the drag.

It is hoped eventually to extend the methods employed here to deal with larger values of R and T and to investigate the formation of the Taylor column. The main object would be to test the discrepancy between experimental results and previous theoretical models of this phenomenon. It has not yet been possible to extend the present work beyond the small range of R and T considered here and at the same time be confident of the reliability of the numerical results. The principal reason for the difficulty is the problem of satisfying adequately the boundary condition at large distances; this is undoubtedly the most sensitive part of the calculation procedure and further developments must await a satisfactory method of dealing with this difficulty.

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